

تمرين 1 :

$$\int_0^1 x \sqrt{x} dx = \int_0^1 x^{\frac{3}{2}} dx = \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^1 = \frac{2}{5}$$

$$\int_0^{\ln(2)} e^{2x} + \frac{1}{e^{3x}} + \sqrt{e^x} dx = \int_0^{\ln(2)} e^{2x} + e^{-3x} + e^{\frac{x}{2}} dx = \left[\frac{1}{2} e^{2x} + \frac{-1}{3} e^{-3x} + 2e^{\frac{x}{2}} \right]_0^{\ln(2)}$$

$$= \left(2 - \frac{1}{3} \times \frac{1}{8} + 2\sqrt{2} \right) - \left(\frac{1}{2} - \frac{1}{3} + 2 \right) = \frac{-1}{24} + 2\sqrt{2} - \frac{1}{6} = 2\sqrt{2} - \frac{5}{24}$$

$$\int_1^2 \frac{1+x}{x^2} dx = \int_1^2 \frac{1}{x^2} + \frac{1}{x} dx = \left[\frac{-1}{x} + \ln(x) \right]_1^2 = \left(\frac{-1}{2} + \ln(2) \right) - (-1) = \frac{1}{2} + \ln(2)$$

$$\int_1^2 \frac{x^2}{1+x} dx = \int_1^2 \frac{x^2 - 1 + 1}{1+x} dx = \int_1^2 x - 1 - \frac{1}{1+x} dx = \left[\frac{1}{2} x^2 - x - \ln(x+1) \right]_1^2 = (2 - 2 - \ln(3)) - \left(\frac{1}{2} - 1 - \ln(2) \right)$$

$$= \ln\left(\frac{2}{3}\right) + \frac{1}{2}$$

$$\int_0^{\frac{f}{4}} \tan x dx = \int_0^{\frac{f}{4}} \frac{\sin x}{\cos x} dx = -\int_0^{\frac{f}{4}} \frac{-\sin x}{\cos x} dx = -\left[\ln|\cos x| \right]_0^{\frac{f}{4}} = -\ln\left(\frac{1}{\sqrt{2}}\right) = \frac{\ln(2)}{2}$$

$$\int_0^{\frac{f}{4}} \tan^2 x dx = \int_0^{\frac{f}{4}} \tan^2 x + 1 - 1 dx = \left[\tan x - x \right]_0^{\frac{f}{4}} = 1 - \frac{f}{4}$$

$$\int_2^3 \frac{2}{x^2 - 1} dx = \int_2^3 \frac{(x+1) - (x-1)}{x^2 - 1} dx = \int_2^3 \frac{1}{x-1} - \frac{1}{x+1} dx = \left[\ln|x-1| - \ln|x+1| \right]_2^3$$

$$= (\ln(2) - \ln(4)) - (\ln(1) - \ln(3)) = \ln\left(\frac{3}{2}\right)$$

$$\int_1^e \frac{\ln(x)}{x} dx = \left[\frac{1}{2} \ln^2(x) \right]_1^e = \frac{1}{2}$$

$$\int_0^2 \frac{1}{|x-1|+1} dx = \int_0^1 \frac{1}{1-x+1} dx + \int_1^2 \frac{1}{x-1+1} dx = \int_0^1 \frac{-1}{x-2} dx + \int_1^2 \frac{1}{x} dx = \left[-\ln|x-2| \right]_0^1 + \left[\ln|x| \right]_1^2$$

$$= \ln 2 + \ln 2 = 2 \ln 2$$

$$\int_0^{\ln(2)} \frac{1}{e^x + 1} dx = \int_0^{\ln(2)} \frac{1 + e^x - e^x}{e^x + 1} dx = \int_0^{\ln(2)} 1 - \frac{e^x}{e^x + 1} dx = \left[x - \ln(e^x + 1) \right]_0^{\ln(2)} = \ln(2) - \ln(3) + \ln(2) = \ln\left(\frac{4}{3}\right)$$

$$\int_e^{e^2} \frac{1}{x \ln(x)} dx = \int_e^{e^2} \frac{\frac{1}{x}}{\ln(x)} dx = \left[\ln(\ln x) \right]_e^{e^2} = \ln(2)$$

$$\int_0^{\frac{f}{2}} \sin^5(x) dx = \int_0^{\frac{f}{2}} (\sin^2(x))^2 \sin(x) dx = \int_0^{\frac{f}{2}} (1 - \cos^2(x))^2 \sin(x) dx$$

$$= \int_0^{\frac{f}{2}} \sin x - 2 \cos^2 x \sin x + \cos^4 x \sin x dx$$

$$\int_0^{\frac{f}{2}} \sin^5(x) dx = \left[-\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x \right]_0^{\frac{f}{2}} = -\left(-1 + \frac{2}{3} - \frac{1}{5} \right) = \frac{8}{15}$$

$$\int_0^1 \frac{x^3}{x^2+1} dx = \int_0^1 \frac{x^3+x-x}{x^2+1} dx = \int_0^1 x - \frac{x}{x^2+1} dx = \left[\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) \right]_0^1 = \frac{1}{2} - \frac{\ln(2)}{2}$$

$$\begin{aligned} \int_2^3 \frac{1}{x^4-1} dx &= \frac{1}{2} \int_2^3 \frac{(x^2+1)-(x^2-1)}{(x^2-1)(x^2+1)} dx = \frac{1}{2} \int_2^3 \frac{1}{x^2-1} - \frac{1}{x^2+1} dx = \frac{1}{2} \int_2^3 \frac{1}{2} \frac{(x+1)-(x-1)}{(x-1)(x+1)} - \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \int_2^3 \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1} - \frac{1}{x^2+1} dx = \frac{1}{2} \left[\frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| - \text{Arctan } x \right]_2^3 \\ &= \frac{1}{2} \left[\frac{1}{2} \ln \left(\frac{|x-1|}{|x+1|} \right) - \text{Arctan } x \right]_2^3 = \frac{1}{2} \left(\frac{1}{2} \ln \left(\frac{1}{2} \right) - \text{Arctan } 3 + \text{Arctan } 2 \right) \end{aligned}$$

$$\int_2^3 \frac{1}{x^4-1} dx = \frac{-\ln(2)}{4} + \frac{\text{Arctan } 2 - \text{Arctan } 3}{2}$$

$$\int_0^1 \frac{\text{Arctan } x}{x^2+1} dx = \int_0^1 \text{Arctan } x \times \frac{1}{x^2+1} dx = \left[\frac{1}{2} \text{Arctan}^2 x \right]_0^1 = \frac{f^2}{32}$$

$$\int_1^2 \frac{1}{\sqrt{x}(x+1)} dx = \int_1^2 2 \frac{\frac{1}{2\sqrt{x}}}{\left(1+(\sqrt{x})^2\right)} dx = \left[2 \text{Arctan } \sqrt{x} \right]_1^2 = 2 \text{Arctan } \sqrt{2} - \frac{f}{2}$$

تمرين 2 :

نضع : $t = x + 2$ منه : $x = t - 2$ و $dx = dt$ و $x = 0 \Rightarrow t = 2$ و $x = 1 \Rightarrow t = 3$ منه :

$$\int_0^1 \frac{x}{\sqrt{x+2}} dx = \int_2^3 \frac{t-2}{\sqrt{t}} dt = \int_2^3 \frac{t}{\sqrt{t}} - \frac{2}{\sqrt{t}} dt = \int_2^3 t^{\frac{1}{2}} - 2t^{-\frac{1}{2}} dt = \left[\frac{2}{3}t^{\frac{3}{2}} - 4t^{\frac{1}{2}} \right]_2^3$$

$$\int_0^1 \frac{x}{\sqrt{x+2}} dx = \frac{2}{3} \times 3\sqrt{3} - 4\sqrt{3} - \frac{2}{3} \times 2\sqrt{2} + 2\sqrt{2} = -2\sqrt{3} + \frac{2}{3}\sqrt{2}$$

نضع : $t = 1 - x$ منه : $x = 1 - t$ و $dx = -dt$ و $x = 0 \Rightarrow t = 1$ و $x = 1 \Rightarrow t = 0$ منه :

$$\int_0^1 x(1-x)^{2015} dx = \int_1^0 -(1-t)t^{2015} dt = \int_0^1 (1-t)t^{2015} dt = \int_0^1 t^{2015} - t^{2016} dt = \left[\frac{t^{2015}}{2015} - \frac{t^{2016}}{2016} \right]_0^1$$

$$\int_0^1 x(1-x)^{2015} dx = \frac{1}{2015 \times 2016} = \frac{1}{4062240}$$

نضع : $t = \frac{2x+1}{\sqrt{3}}$ منه : $x = \frac{t\sqrt{3}-1}{2}$ و $dx = \frac{\sqrt{3}}{2} dt$ و $x = 0 \Rightarrow t = \frac{1}{\sqrt{3}}$ و $x = 1 \Rightarrow t = \sqrt{3}$ منه :

$$\int_0^1 \frac{1}{x^2+x+1} dx = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{\frac{3t^2-2\sqrt{3}+1}{4} + \frac{t\sqrt{3}-1}{2} + 1} \frac{\sqrt{3}}{2} dt = \frac{\sqrt{3}}{2} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{4}{3t^2-2\sqrt{3}t+1+2\sqrt{3}t-2+4} dt$$

$$\int_0^1 \frac{1}{x^2+x+1} dx = \frac{\sqrt{3}}{2} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{4}{3t^2+3} dt = \frac{2\sqrt{3}}{3} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{t^2+1} dt = \frac{2\sqrt{3}}{3} [\text{Arctan } t]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{2\sqrt{3}}{3} \left(\frac{f}{3} - \frac{f}{6} \right) = \frac{\sqrt{3}f}{9}$$

لا يصح أن نكتب : $dx = \frac{\sqrt{3} dt}{2}$ مثل المتجهات لا يصح أن نكتب مثلا $\vec{AB} = \frac{\sqrt{3}\vec{AC}}{2}$

نضع : $t = \sqrt{e^x+1}$ منه : $x = \ln(t^2-1)$ و $dx = \frac{2t}{t^2-1} dt$ و $x = 0 \Rightarrow t = \sqrt{2}$ و $x = 1 \Rightarrow t = \sqrt{e+1}$ منه :

$$\int_0^1 \frac{e^{2x}}{\sqrt{e^x+1}} dx = \int_{\sqrt{2}}^{\sqrt{e+1}} \frac{(t^2-1)^2}{t} \frac{2t}{t^2-1} dt = 2 \int_{\sqrt{2}}^{\sqrt{e+1}} t^2 - 1 dt = 2 \left[\frac{t^3}{3} - t \right]_{\sqrt{2}}^{\sqrt{e+1}}$$

$$\int_0^1 \frac{e^{2x}}{\sqrt{e^x+1}} dx = 2 \left(\frac{(e+1)\sqrt{e+1}}{3} - \sqrt{e+1} - \frac{2\sqrt{2}}{3} + \sqrt{2} \right) = \frac{2}{3} ((e-2)\sqrt{e+1} + \sqrt{2})$$

نضع : $t = \sqrt{\sqrt{x+1}}$ منه : $x = (t^2 - 1)^2$ و $dx = 4t(t^2 - 1)dt$ و $x = 0 \Rightarrow t = 1$ منه : $x = 1 \Rightarrow t = \sqrt{2}$

$$\int_0^1 \sqrt{\sqrt{x+1}} dx = \int_1^{\sqrt{2}} 4t^2(t^2 - 1) dx = 4 \int_1^{\sqrt{2}} t^4 - t^2 dx = 4 \left[\frac{t^5}{5} - \frac{t^3}{3} \right]_1^{\sqrt{2}} = 4 \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3} \right) = \frac{8}{15} (\sqrt{2} + 1)$$

نعلم أن الدالة $x \rightarrow \sin x$ متصلة و تزايدية قطعاً على $\left[0; \frac{f}{2}\right]$ ، إذن فهي تقبل دالة عكسية $\{x\}$ معرفة

من $[0;1]$ نحو $\left[0; \frac{f}{2}\right]$ نضع : $t = \{x\}$ منه : $x = \sin t$ و $dx = \cos t dt$

$$x = 0 \Rightarrow t = 0$$

$$: \text{ منه } x = 1 \Rightarrow t = \frac{f}{2} \text{ و } \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t$$

$$\int_0^1 \sqrt{1-x^2} dx = \int_0^{\frac{f}{2}} \cos^2 t dt = \int_0^{\frac{f}{2}} \frac{1+\cos 2t}{2} dt = \frac{1}{2} \left[t + \frac{\sin(2t)}{2} \right]_0^{\frac{f}{2}} = \frac{f}{4}$$

لا يصح أن نضع $x = \sin t$ مباشرة (رغم أنها إشارة في السؤال) ، لأن الوضع يتم عن طريق دالة للمتغير x ، فطريقة تغيير المتغير مجرد اختصار لخاصية تكامل دالة باستعمال مركب دالتين، والتي تخضع لشروط يجب مراعاتها

تمرين 3 :

$$\int_0^1 (3+2x)e^x dx = \int_0^1 (3+2x)(e^x)' dx = [(3+2x)e^x]_0^1 - \int_0^1 (3+2x)' e^x dx = [(3+2x)e^x]_0^1 - \int_0^1 2e^x dx$$

$$\int_0^1 (3+2x)e^x dx = [(3+2x)e^x - 2e^x]_0^1 = [(1+2x)e^x]_0^1 = 3e - 1$$

$$\int_1^2 \ln(x) dx = \int_1^2 x' \ln(x) dx = [x \ln(x)]_1^2 - \int_1^2 x (\ln(x))' dx = [x \ln(x)]_1^2 - \int_1^2 1 dx = [x \ln(x) - x]_1^2$$

$$\int_1^2 \ln(x) dx = 2 \ln 2 - 2 + 1 = 2 \ln 2 - 1$$

$$\int_0^{\frac{f}{2}} x \cos(x) dx = \int_0^{\frac{f}{2}} x (\sin(x))' dx = [x \sin(x)]_0^{\frac{f}{2}} - \int_0^{\frac{f}{2}} x' \sin(x) dx = [x \sin(x)]_0^{\frac{f}{2}} - \int_0^{\frac{f}{2}} \sin(x) dx$$

$$\int_0^{\frac{f}{2}} x \cos(x) dx = [x \sin(x) + \cos(x)]_0^{\frac{f}{2}} = \frac{f}{2} - 1$$

$$\int_1^2 x \ln(x) dx = \int_1^2 \left(\frac{x^2}{2} \right)' \ln(x) dx = \left[\frac{x^2}{2} \ln(x) \right]_1^2 - \int_1^2 \frac{x^2}{2} (\ln(x))' dx = \left[\frac{x^2}{2} \ln(x) \right]_1^2 - \int_1^2 \frac{x}{2} dx$$

$$\int_1^2 x \ln(x) dx = \left[\frac{x^2}{2} \ln(x) - \frac{x^2}{4} \right]_1^2 = 2 \ln 2 - 1 + \frac{1}{4} = 2 \ln 2 - \frac{3}{4}$$

$$\int_1^e \frac{\ln(x)}{x^2} dx = \int_1^e \left(-\frac{1}{x} \right)' \ln(x) dx = \left[\frac{-\ln(x)}{x} \right]_1^e - \int_1^e \frac{-1}{x} (\ln(x))' dx = \left[\frac{-\ln(x)}{x} \right]_1^e + \int_1^e \frac{1}{x^2} dx$$

$$\int_1^e \frac{\ln(x)}{x^2} dx = \left[\frac{-\ln(x)}{x} - \frac{1}{x} \right]_1^e = \frac{-1}{e} - \frac{1}{e} + 1 = \frac{e-2}{e}$$

$$\begin{aligned} \int_1^2 x \arctan x \, dx &= \int_1^2 \left(\frac{x^2}{2} \right)' \arctan x \, dx = \left[\frac{x^2}{2} \arctan x \right]_1^2 - \int_1^2 \frac{x^2}{2} (\arctan x)' \, dx \\ &= \left[\frac{x^2}{2} \arctan x \right]_1^2 - \frac{1}{2} \int_1^2 \frac{x^2}{x^2+1} \, dx = \left[\frac{x^2}{2} \arctan x \right]_1^2 - \frac{1}{2} \int_1^2 1 - \frac{1}{x^2+1} \, dx \\ &= \left[\frac{x^2}{2} \arctan x - \frac{1}{2}(x - \arctan x) \right]_1^2 = \frac{1}{2} \left[(x^2+1) \arctan x - x \right]_1^2 = \frac{1}{2} \left(5 \arctan 2 - 2 - \frac{f}{2} + 1 \right) \\ \int_1^2 x \arctan x \, dx &= \frac{10 \arctan 2 - 2 - f}{4} \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{f}{2}} \sin x e^x \, dx &= \int_0^{\frac{f}{2}} \sin x (e^x)' \, dx = \left[\sin x e^x \right]_0^{\frac{f}{2}} - \int_0^{\frac{f}{2}} (\sin x)' e^x \, dx = \left[\sin x e^x \right]_0^{\frac{f}{2}} - \int_0^{\frac{f}{2}} \cos x e^x \, dx \\ &= \left[\sin x e^x \right]_0^{\frac{f}{2}} - \int_0^{\frac{f}{2}} \cos x (e^x)' \, dx = \left[\sin x e^x \right]_0^{\frac{f}{2}} - \left(\left[\cos x e^x \right]_0^{\frac{f}{2}} - \int_0^{\frac{f}{2}} (\cos x)' e^x \, dx \right) \\ \int_0^{\frac{f}{2}} \sin x e^x \, dx &= \left[\sin x e^x \right]_0^{\frac{f}{2}} - \left(\left[\cos x e^x \right]_0^{\frac{f}{2}} + \int_0^{\frac{f}{2}} \sin x e^x \, dx \right) = \left[\sin x e^x - \cos x e^x \right]_0^{\frac{f}{2}} - \int_0^{\frac{f}{2}} \sin x e^x \, dx \\ \int_0^{\frac{f}{2}} \sin x e^x \, dx &= \frac{1}{2} \left(e^{\frac{f}{2}} + 1 \right) : \text{منه} \quad 2 \int_0^{\frac{f}{2}} \sin x e^x \, dx = \left[(\sin x - \cos x) e^x \right]_0^{\frac{f}{2}} : \text{منه} \end{aligned}$$

في هذا التكامل الخاص قمنا بالتعبير عن هذا التكامل بدلالة نفسه وبذلك تمكنا من تحديد قيمته

$$\begin{aligned} \int_1^e (\ln x)^2 \, dx &= \int_1^e x' (\ln x)^2 \, dx = \left[x (\ln x)^2 \right]_1^e - \int_1^e x ((\ln x)^2)' \, dx = \left[x (\ln x)^2 \right]_1^e - \int_1^e 2 \ln x \, dx \\ &= \left[x (\ln x)^2 \right]_1^e - 2 \int_1^e x' \ln x \, dx = \left[x (\ln x)^2 \right]_1^e - 2 \left(\left[x \ln x \right]_1^e - \int_1^e x (\ln x)' \, dx \right) \\ \int_1^e (\ln x)^2 \, dx &= \left[x (\ln x)^2 - 2x \ln x \right]_1^e + 2 \int_1^e 1 \, dx = \left[x (\ln x)^2 - 2x \ln x + 2x \right]_1^e = e - 2e + 2e - 2 = e - 2 \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{x+1}{e^x} \, dx &= \int_0^1 (x+1) e^{-x} \, dx = \int_0^1 (x+1) (-e^{-x})' \, dx = \left[-(x+1) e^{-x} \right]_0^1 - \int_0^1 (x+1)' (-e^{-x}) \, dx \\ \int_0^1 \frac{x+1}{e^x} \, dx &= \left[-(x+1) e^{-x} \right]_0^1 + \int_0^1 e^{-x} \, dx = \left[-(x+1) e^{-x} - e^{-x} \right]_0^1 = \left[\frac{-x-2}{e^x} \right]_0^1 = \frac{-3}{e} + 2 = \frac{2e-3}{e} \end{aligned}$$

$$\forall x \in \mathbb{R}_{-\{1,3\}} \quad \frac{-3x^2+7x+2}{x^2-2x-3} = a + \frac{b}{x+1} + \frac{c}{x-3} \quad \text{تمرين 4}$$

$$a + \frac{b}{x+1} + \frac{c}{x-3} = a + \frac{bx-3b+cx+c}{(x+1)(x-3)} = \frac{ax^2-2ax-3a+bx-3b+cx+c}{x^2-2x-3} \quad \text{لدينا}$$

$$\forall x \in \mathbb{R}_{-\{1,3\}} \quad ax^2 + (-2a+b+c)x + (-3a-3b+c) = -3x^2 + 7x + 2 \quad \text{منه}$$

$$\begin{cases} a = -3 \\ c = -1 \\ b = 2 \end{cases} : \text{منه} \quad \begin{cases} a = -3 \\ c = 1-b \\ 4b = 8 \end{cases} : \text{منه} \quad \begin{cases} a = -3 \\ b+c = 1 \\ -3b+c = -7 \end{cases} : \text{منه} \quad \begin{cases} a = -3 \\ -2a+b+c = 7 \\ -3a-3b+c = 2 \end{cases} : \text{منه} \quad 1$$

هناك طرق أخرى للجواب عن هذا السؤال كالقسمة الإقليدية

$$\begin{aligned} I &= \int_0^2 \frac{2-3x^2+7x+2}{x^2-2x-3} \, dx = \int_0^2 -3 + \frac{2}{x+1} - \frac{1}{x-3} \, dx = \left[-3x + 2 \ln|x+1| - \ln|x-3| \right]_0^2 \\ I &= -6 + 2 \ln(3) + \ln(3) = 3 \ln(3) - 6 \quad 2 \end{aligned}$$

تمرين 5

	$\forall x \in \mathbb{R} \quad \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^x(e^x - e^{-x})}{e^x(e^x + e^{-x})} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	1
	$I = \int_0^1 \frac{e^{2t} - 1}{e^{2t} + 1} dt = \int_0^1 \frac{e^t - e^{-t}}{e^t + e^{-t}} dt = \left[\ln(e^t + e^{-t}) \right]_0^1 = \ln\left(\frac{e^2 + 1}{2e}\right)$	2